GENERALIZED BARONTI CONSTANT 
AND NORMAL STRUCTURE

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Abstract

We introduce the definition of generalized Baronti constant \( A_2(a, X) \) in Banach space, analyze some properties of this modulus, and construct the relation between this modulus and other geometrical constants, the main result is: If \( A_2(a, X) < 1 + a \), for some \( a \in [0, 1] \), then Banach space \( X \) has uniform normal structure.

1. Introduction

The properties which can imply metric fixed point theory in a Banach space have been studied widely. Some properties of Jordan-von Neumann constant and James constant have been shown to imply uniform normal structure [2], [4].

Baronti et al. [1] defined parameter \( A_2(X) \) to inscribe normal structure. In this paper, we consider the generalized constants \( A_2(a, X) \).
and give some properties. And discuss the relations through these constants. We show that, if $A_2(a, X) < (1 + a)$ for some $a \in [0, 1]$, then $X$ possesses uniform normal structure. As an example, we compute $A_2(a, X)$, for all $a \in [0, 2]$, when $X$ is a Hilbert space.

2. Preliminaries

Throughout the paper, we let $X$ stand for a Banach space. By $B_X$ and $S_X$, we denote the closed unit ball and the unit sphere of $X$, respectively. We shall say that a nonempty weakly compact convex subset $C$ of $X$ has the fixed point property (fpp for short), if every nonexpansive mapping $T : C \to C$ has a fixed point (that is, there exits $x \in C$ such that $T(x) = x$). Recall that $T$ is nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. We shall say that $X$ has the fixed point property (fpp), if every weak compact convex subset of $X$ has the fpp. Let $A$ be a nonempty bounded set in $X$. The number $r(A) = \inf \{ \sup_{y \in A} \|x - y\| : x \in A \}$ is called the Chebyshev radius of $A$. The number $\text{diam } A = \sup \{ \|x - y\| : x, y \in A \}$ is called the diameter of $A$. A Banach space $X$ has normal structure, if

$$r(A) < \text{diam } A,$$

(2.1)

for every bounded convex closed subset $A$ of $X$ with $\text{diam } A > 0$. When (1.1) holds for every weakly compact convex subset $A$ of $X$ with $\text{diam } A > 0$, we say $X$ has weak normal structure. Normal structure and weak normal structure coincide, if $X$ is reflexive. A space $X$ is said to have uniform normal structure, if $\inf(\{ \text{diam } A / (r(A)) \}) > 1$, where the infimum is taken over all bounded convex closed subsets $A$ of $X$ with $\text{diam } A > 0$. Weak structure, as well as many other properties imply the fpp. The relevant papers are [8], [9], [10], and so on.

The modulus of convexity of $X$ is a function $\delta_X : [0, 2] \to [0, 2]$ defined by $\delta_X(\varepsilon) = \inf \{ 1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| \geq \varepsilon \}$. If $\delta_X(1) \geq 0$, then $X$ has uniform normal structure [2], [5].
Definition 2.1 [10]. Let $U$ be a filter on $I$, then $\{x_i\}$ is said to convergence $x$ with respect to $U$, denoted by $\lim_U x_i = x$, if for each neighborhood $V$ of $x$, $\{i \in I : x_i \in V\} \in U$. $U$ is called an ultrafilter, if it is maximal with respect to the ordering of set inclusion. An ultrafilter is called trivial, if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$.

We will use the fact that: if $U$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \in U$, or $I \setminus A \in U$;
(ii) if $\{x_i\}$ has a cluster point $x$, then $\lim_U x_i = x$.

Let $\{X_i\}$ be a family of Banach spaces, and let $l_\omega(I, X_i)$ denote the subspace of the product space $\prod X_i$, equipped with the norm $\|x_i\| = \sup_{i \in I} \|x_i\| < \infty$. Let $U$ be an ultrafilter on $I$, $N_U = \{(x_i) \in l_\omega(I, X_i) : \lim_U \|x_i\| = 0\}$. The ultraproduct of $\{X_i\}_{i \in I}$ is the quotient space $l_\omega(I, X_i) / N_U$. We will use $\tilde{x}$ to denote the element of the ultraproduct. It follows from property (ii) above and the definition of quotient norm that,

$$\|\tilde{x}\| = \lim_{U} \|x_i\|.$$ 

In what follows, we will restrict our set $I$ to be $\mathbb{N}$, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach spaces $X$. For an ultrafilter $U$ on $\mathbb{N}$, we use $\tilde{X}$ denote the ultraproduct. It is also clear that $X$ is isometric to subspace of $\tilde{X}$. Hence, we may assume that $X$ is a subspace of $\tilde{X}$.

Lemma 2.1. If $X$ is a Banach space, then $(\tilde{X})^* = (\tilde{X}^*)$, iff $X$ is super-reflexive.

3. Main Results

Definition 3.2. Let $X$ be a Banach space, for $a \geq 0$,

$$A_2(a, X) = \sup \left\{ \frac{\|x + y\| + \|x - z\|}{2} : \|y - z\| \leq a\|x\|, x, y, z \in B_X \right\}.$$
of which at least one belongs to $S_X$.

First let us show some clear properties of $A_2(a, X)$:

1. $A_2(0, X) = A_2(x)$;
2. $A_2(a, X)$ is a nondecreasing function with respect to $a$;
3. $1 + \frac{a}{2} \leq A_2(a, X) \leq 2$, $a \in [0, 2]$;
4. If $A_2(a, X) < 2$, for some $a \geq 0$, then $A_2(X) < 2$ and consequently, $X$ is uniformly nonsquare.

**Theorem 3.2.** For a Hilbert space $H$, $A_2(a, H) = \sqrt{2 + a}$.

**Proof.** Let $x, y, z \in B_H$ with $\|y - z\| \leq a\|x\|$. On one hand, we have

$$\|x + y\| + \|x - z\| \leq \sqrt{\frac{\|x + y\|^2 + \|x - z\|^2}{2}}$$

$$= \sqrt{\frac{2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2 <x, y - z>} \frac{1}{2}}$$

$$\leq \sqrt{\frac{4 + 2\|x\|\|y - z\|}{2}}$$

$$\leq \sqrt{2 + a}.$$

On the other hand, let $e_1$ and $e_2$ be orthonormal elements of $S_H$. Put

$$x = e_1, \ y = \frac{a}{2}e_1 + \sqrt{1 - \frac{a^2}{4}}e_2, \ z = -\frac{a}{2}e_1 + \sqrt{1 - \frac{a^2}{4}}e_2.$$ 

Thus, we have $\|y - z\| = a\|x\|$ and $\frac{\|x + y\| + \|x - z\|}{2} = \sqrt{2 + a}$.

**Theorem 3.3.** For a Banach space $X$, $\frac{A_2(a, X)^2}{2} \leq C_{NJ}(a, X)$ for all $a \in [0, \infty)$.
Lemma 3.4 [4]. Let $X$ be a Banach space. For $0 \leq a < 2$, if $C_{NJ}(a, X) = 2$, then there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in $B_X$ satisfying:

1. $\|x_n\|, \|y_n\|, \|z_n\| \to 1$;
2. $\|x_n + y_n\|, \|x_n - z_n\| \to 2$;
3. $\|y_n - z_n\| \leq a\|x\|$ for all $n$.

Furthermore, the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ can be chosen from $S_X$.

Corollary 3.5. For a Banach space $X$, $A_2(a, X) = 2$, if and only if $C_{NJ}(a, X) = 2$ for all $a \in [0, 2]$.

Proof. If $C_{NJ}(a, X) = 2$, then by lemma, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in $S_X$ satisfying $\|x_n + y_n\|, \|x_n - z_n\| \to 2$, and $\|y_n - z_n\| \leq a$ for all $n$. Thus, $A_2(a, X) = 2$. The other direction is an easy consequence of proposition.

Corollary 3.6. For a Banach space $X$, $J(a, X) = 2$, if and only if $A_2(a, X) = 2$.

Corollary 3.7. Let $X$ be a Banach space. If $J(1, X) < 2$ or $A_2(a, X) < 2$, then $X$ has uniform normal structure.

Proposition 3.1. For $0 \leq a \leq b$, $A_2(b, X) + \frac{a}{2} \leq A_2(a, X) + \frac{b}{2}$. In particular, $A_2(\cdot, X)$ is continuous on $[0, \infty)$.

Proof. Let $\varepsilon > 0$. There exist $x, y, z \in B_X$ such that $\|y - z\| = b_1\|x\|$, and $A_2(b, X) - \varepsilon \leq \frac{\|x + y\| + \|x - z\|}{2}$, $b_1$ can be chosen so that $a < b_1$. Otherwise, the assertion is obviously true. We can choose $z_1, y_1 \in B_X$ such that $\|y - y_1\|, \|z - z_1\| \leq \frac{b - a}{2}$, and $\|y_1 - z_1\| \leq a\|x\|$. Then, we have

$$A_2(b, X) - \varepsilon \leq \frac{\|x + y\| + \|x - z\|}{2}$$
To finish the proof, we let $\varepsilon \to 0$.

**Lemma 3.8.** Uniformly nonsquare Banach spaces are super-reflexive.

**Corollary 3.9.** $A_2(a, X) = A_2(a, \tilde{X})$.

**Proof.** Clearly, $A_2(a, X) \leq A_2(a, \tilde{X})$. To show $A_2(a, X) \geq A_2(a, \tilde{X})$, let $\delta > 0$, $\alpha \in [0, a]$ and suppose $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$ and $\|\tilde{y} - \tilde{z}\| = \alpha \|\tilde{x}\|$. If $\tilde{x} = 0$, then $\frac{\|\tilde{x}\| + \|\tilde{z}\|}{2} = \|\tilde{y}\| \leq 1 \leq A_2(a, X)$. If $\tilde{x} \neq 0$, choose $\varepsilon > 0$ such that $\varepsilon < \delta \|\tilde{x}\|$. Since

$$c := \frac{\|\tilde{x} + \tilde{y}\| + \|\tilde{x} - \tilde{z}\|}{2} = \lim_{U} \frac{\|x_n + y_n\| + \|x_n - z_n\|}{2} := \lim_{U} c_n,$$

the set $\{n \in N : |c_n - c| < \delta$ and $\|y_n - z_n\| \leq \alpha \|x_n\| + \varepsilon < (\alpha + \delta) \|x_n\|\}$ belongs to $U$.

In particular,

$$c \leq \frac{\|x_n + y_n\| + \|x_n - z_n\|}{2} + \delta \leq A_2(a + \delta, X) + \delta$$

for some $n$.

Then, the inequality $A_2(a, X) \geq A_2(a, \tilde{X})$ follows from the arbitrariness of $\delta$ and the continuity of $A_2(\cdot, X)$.

**Lemma 3.10.** Let $X$ be a Banach space without weak normal structure, then for any $0 < \varepsilon < 1$ and each $0 \leq t \leq 1$, there exist $x_1 \in S_X$, $x_2, x_3 \in tS_X$ satisfying:

1. $x_2 - x_3 = ax_1$ with $|a - t| < \varepsilon$;
2. $\|x_1 + x_2\| > (1 + t) - 3\varepsilon$, $\|x_1 + (x_3 - x_2)\| > (1 + t) - 3\varepsilon$. 


Theorem 3.11. Let $X$ be a Banach space. If $A_2(a, X) < 1 + a$, for some $a \in [0, 1]$, then $X$ has uniform normal structure.

Proof. It suffices to show that these conditions imply $X$ has normal structure. For the case $A_2(a, X) < 1 + a$, $a \in [0, 1]$, and Remark 3.1, $X$ is uniformly nonsquare and so in turn is reflexive. Thus normal structure and weak normal structure coincide, it suffices to prove that $X$ has weak normal structure. By the continuity of $A_2(\cdot, X)$, $A_2(a', X) < 1 + a$, for some $a' > a$. Choose $m \in N$ such that $a + \frac{1}{m} \leq a'$. Suppose $X$ does not have weak normal structure, by lemma, there exist $x_n \in S_X$, $y_n, z_n \in aS_X$ such that, for each $n \in N$, $y_n - z_n = a_n x_n$ with $|a_n - a| < \frac{1}{n + m}$, $\|x_n + y_n\| > (1 + a) - \frac{3}{n + m}$, $\|x_n - z_n\| > (1 + t) - \frac{3}{n + m}$.

$\|y_n - z_n\| = a_n < a + \frac{1}{n + m} \leq a'$ and $\liminf_{n \to \infty} \|x_n + y_n\| \geq 1 + a$, and\n
$\liminf_{n \to \infty} \|x_n - z_n\| \geq 1 + a$. Thus,

$$1 + a \leq \liminf_{n \to \infty} \frac{\|x_n + y_n\| + \|x_n - z_n\|}{2} \leq A_2(a', X) < 1 + a.$$\n
This contradiction shows that $X$ must have weak normal structure.

Corollary 3.12. Let $X$ be a Banach space. If $A_2(1, X) < 2$, then $X$ has uniform normal structure.

Theorem 3.13. Let $X$ be a Banach space, $\varepsilon \in [0, 2]$, and $\beta \geq 0$. If $A_2(a, X) \leq \frac{2 + |\varepsilon - \beta|}{2}$, then $\delta_X(\varepsilon) \geq 0$.

Proof. Suppose $\delta_X(\varepsilon) = 0$, there exist $x_n, y_n \in S_X$, such that $\|x_n - y_n\| = \varepsilon$ for all $n \in N$, and $\liminf_{n \to \infty} \|x_n + y_n\| = 2$. Put $z_n = y_n - \beta x_n$. Then, for each $n \in N$, $y_n - z_n = \beta x_n$, $\|z_n\| = \|y_n - \beta x_n\| \leq 1 + \beta$, and $\|x_n - z_n\| \geq \|x_n - y_n - \beta x_n\| = |\varepsilon - \beta|$. Thus,

$$\frac{2 + |\varepsilon - \beta|}{2} \leq \liminf_{n \to \infty} \frac{\|x_n + y_n\| + \|x_n - z_n\|}{2} \leq \frac{2 + |\varepsilon - \beta|}{2}.$$\n
We obtain a contradiction.
Corollary 3.14. If \( A_2(0, X) < 1 + \frac{\sqrt{3}}{2} \), then \( \delta_X(r) \geq 0 \).

Corollary 3.15. If \( A_2(\cdot, X) \) is concave and \( A_2(a, X) < \frac{\sqrt{3} + 1 + (3 - \sqrt{3})a}{2} \) for some \( a \in [0, 1] \), then \( X \) has uniform normal structure.

Proof. If \( A_2(1, X) < 2 \), we are done by Corollary 3.5. Let \( A_2(1, X) = 2 \) and suppose that \( X \) does not have uniform normal structure. Therefore, \( A_2(0, X) \geq \frac{1 + \sqrt{3}}{2} \). By the concavity of \( A_2(\cdot, X) \), we have, for all \( a \in [0, 1] \)

\[
A_2(a, X) \geq (1 - a)A_2(0, X) + aA_2(1, X) \geq \frac{\sqrt{3} + 1 + (3 - \sqrt{3})a}{2},
\]
a contradiction.

Example 3.16 \((l_\infty - l_1 \text{norm})\). Let \( X = R^2 \) be equipped with the norm defined by

\[
\|x\| = \begin{cases} 
\|x\|_\infty & \text{if } x_1x_2 \geq 0, \\
\|x\|_1 & \text{if } x_1x_2 \leq 0.
\end{cases}
\]

Take \( x = (1, 1) \), \( y = (0, 1) \) and \( z = (-1, 0) \). Then, we have \( y - z = (1, 1) = x \) and \( \|x + y\| = \|(1, 2)\|_\infty = 2 \), \( \|x - z\| = \|(2, 1)\|_\infty = 2 \). So,

\[
2 = (2 + 2) / 2 = \frac{\|x + y\| + \|x - z\|}{2} \leq A_2(1, X) \leq 2.
\]
Hence, \( A_2(1, X) = 2 \).

References


